Zeta Functions of Higher Order and Their Applications

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1 Absolute Tensor Product

In this note we present our results on multiple zeta functions with some applications. This is a survey of our papers [KK1, KK2, KK3, KK4]. We also refer to [KK5, KK6] for applications more recently proved.

Definition 1 (regularized product) Let $m(\rho) \in \mathbb{Z}$ $(\rho \in \mathbb{C})$ be the multiplicity of zeros (or poles) at $s = \rho$ of some meromorphic function $Z(s)$. We define the regularized product as follows:

$$\prod_{\rho \in \mathbb{C}} (s-\rho)^{m(\rho)} := \exp\left(-\frac{\partial}{\partial w}\bigg|_{w=0} \sum_{\rho \in \mathbb{C}} \frac{m(\rho)}{(s-\rho)^w} \right)$$

in case the series in the right hand side converges in $\mathrm{Re}(w) \gg 0$ and has an analytic continuation to $w = 0$.

The absolute tensor product is defined as follows:

Definition 2 (absolute tensor product) The absolute tensor product of zeta functions

$$Z_j(s) = \prod_{\rho \in \mathbb{C}} (s-\rho)^{m_j(\rho)} \qquad\qquad (j = 1, ..., r)$$

is defined by

$$(Z_1 \otimes \cdots \otimes Z_r)(s) := \prod_{\rho \in \mathbb{C}} \big(s - (\rho_1 + \cdots + \rho_r)\big)^{m(\rho_1, \ldots, \rho_r)},$$

where

$$m(\rho_1, ..., \rho_r) = m(\rho_1) \cdots m(\rho_r) \times \begin{cases} 1 & \text{if } \operatorname{Im}(\rho_j) \geq 0 \ (\forall j) \\ (-1)^{r-1} & \text{if } \operatorname{Im}(\rho_j) < 0 \ (\forall j) \\ 0 & \text{otherwise.} \end{cases}$$

For the background and the motivation of this definition, we refer to the excellent survey of Manin [M], where the tensor product is named the Kurokawa product by him.

We introduce the Selberg zeta function for a Riemannian manifold. Let M be a Riemannian manifold, and P be the set of prime closed geodesics. The Selberg zeta function of M is defined as follows as long as P is a countable set and the following Euler product converges:

Definition 3 (Selberg zeta function) We define

$$\zeta_M(s) := \prod_{p \in P} (1 - e^{-l(p)s})^{-1},$$

where $l(p)$ is the length of a geodesic p.

Examples 4 Let $M = S^1(\frac{l}{2\pi})$ be the circle with radius $\frac{l}{2\pi}$. Then P consists of one element which we denote by p. Then

$$\zeta_M(s) = (1 - e^{-l(p)s})^{-1}.$$

Especially when $l(p) = \log q$ with q a power of some prime number, it follows that $\zeta_M(s) = (1 - q^{-s})^{-1} = \zeta(s, \mathbf{F}_q)$ which is the Hasse zeta function of the finite field $\mathbf{F}_q$.

In what follows we denote by p either a prime number or a prime geodesic. The norm of p is defined by

$$N(p) = \begin{cases} e^{l(p)} & (p \in P) \\ p & (p\text{: a prime number}) \end{cases}.$$

Here we introduce the notion of *generic* for real numbers.

Definition 5 (generic) A real number α is called *generic* if and only if

$$\lim_{m \to \infty} \|m\alpha\|^{\frac{1}{m}} = 1,$$

where $\|x\| := \min\{|x - n| \ : \ n \in \mathbb{Z}\}$ for $x \in \mathbb{R}$.

Examples 6

(1) If $\alpha \in \mathbb{Q}$, then α is not generic.

(2) If $\alpha \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \setminus \mathbb{Q}$, then α is generic.

(3) Let $x, y \in \overline{\mathbb{Q}} \cap \mathbb{R}_{>0}$, $y \neq 1$. If $\alpha = \dfrac{\log x}{\log y} \notin \mathbb{Q}$, then α is generic.

The last example was proved by Baker in his famous work on transcendental numbers. We recall it as follows:

Baker's Theorem. Let $x, y \in \overline{\mathbb{Q}}$ and assume that $\frac{\log x}{\log y} \notin \mathbb{Q}$. Then for any $m, n \in \mathbb{Z}$, $m > 0$,

$$\left| m \frac{\log x}{\log y} - n \right| > m^{-c}$$

with c depending only on x and y.

Here we calculate the absolute tensor product for Selberg zeta functions for circles.

Theorem 7 *The absolute tensor product of*

$$Z_j(s) = (1 - e^{-l_j s})^{-1} \qquad (j = 1, 2)$$

is expressed as follows in $\mathrm{Re}(s) > 0$ *with some polynomials* $Q(s)$*:*

(1) When both $\dfrac{l_1}{l_2}$ *and* $\dfrac{l_1}{l_2}$ *are generic,*

$$\begin{aligned} (Z_1 \otimes Z_2)(s) &= e^{Q(s)} \left(1 - e^{-sl_1}\right)^{\frac{1}{2}} \left(1 - e^{-sl_2}\right)^{\frac{1}{2}} \\ &\times \exp\left(\frac{1}{2i} \sum_{k=1}^{\infty} \frac{\cot\left(\pi k \frac{l_1}{l_2}\right)}{k} e^{-l_1 k s} + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{l_2}{l_1}\right)}{n} e^{-l_2 n s} \right). \end{aligned}$$

(2) When $l_1 = l_2 = l$*,*

$$(Z_1 \otimes Z_2)(s) = e^{Q(s)} \left(1 - e^{-ls}\right)^{1 - \frac{ils}{2\pi}} \exp\left(\frac{-1}{2\pi i} \sum_{n=1}^{\infty} \frac{e^{-nls}}{n^2} \right).$$

In particular when $l_1 = \log p$ and $l_2 = \log q$ with some prime powers p and q, the following theorem holds:

Theorem 8 ([KK2]) *Let $\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1}$. We have the following expressions in* $\mathrm{Re}(s) > 0$ *with some polynomials $Q(s)$.*

(1) When $p \neq q$,

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) = e^{Q(s)} \left(1 - p^{-s}\right)^{\frac{1}{2}} \left(1 - q^{-s}\right)^{\frac{1}{2}}$$
$$\times \exp\left(\frac{1}{2i} \sum_{k=1}^{\infty} \frac{\cot\left(\pi k \frac{\log p}{\log q}\right)}{k} p^{-ks} + \frac{1}{2i} \sum_{n=1}^{\infty} \frac{\cot\left(\pi n \frac{\log q}{\log p}\right)}{n} q^{-ns} \right).$$

(2) When $p = q$,

$$\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_p) = e^{Q(s)} \left(1 - p^{-s}\right)^{1 - \frac{is \log p}{2\pi}} \exp\left(\frac{-1}{2\pi i} \sum_{n=1}^{\infty} \frac{p^{-ns}}{n^2} \right).$$

Remark 9 (Convergence) The convergence of the power series in the right hand side of Theorems 7(1) and 8(1) is subtle. When $\alpha \in \mathbb{R}$ is generic, we deduce from the definition that $|m\alpha - n| > e^{-\varepsilon m}$ for any $m \geq 1$ and any $n \in \mathbb{Z}$. Thus it holds that $\cot(\pi m \alpha) = O(e^{\varepsilon m})$ for any $\varepsilon > 0$. Hence the series

$$\sum_{m=1}^{\infty} \cot(\pi m \alpha) x^m$$

absolutely converges in $|x| < 1$. This is the reason why we need the assumption of genericity. In Theorem 8 we do not need the assumption with help of the Baker's theorem. When $\alpha = \frac{\log p}{\log q}$, the Baker's theorem leads to $|m\alpha - n| > m^{-c}$ for any $m \geq 1$ and $n \in \mathbb{Z}$. Then $\cot(\pi m \alpha) = O(m^c)$ and hence the series again absolutely converges in $|x| < 1$.

Remark 10 (Euler product) Assume Z_j has an analytic continuation, a functional equation and an Euler product expression

$$Z_j(s) = \prod_{p} H_p^{(j)}(N(p)^{-s})$$

in $\mathrm{Re}(s) > \sigma_j$ with $H_p^{(j)}(T) \in 1 + T\mathbb{C}[[T]]$. Then $Z_1 \otimes \cdots \otimes Z_r$ would have an Euler product

$$(Z_1 \otimes \cdots \otimes Z_r)(s) = e^{Q(s)} \prod_{p_1,\dots,p_r} H_{p_1,\dots,p_r}(N(p_1)^{-s}, \dots, N(p_r)^{-s})$$

with $H_{p_1,\dots,p_r}(T_1,\dots,T_r) \in 1 + (T_1,\dots,T_r)\mathbb{C}[[T_1,\dots,T_r]]$ and some polynomial $Q(s)$. Theorem 8 gives an example of this fact where we put $H_p^{(1)}(p^{-s}) = (1-p^{-s})^{-1}$, $H_q^{(2)}(q^{-s}) = (1-q^{-s})^{-1}$ and the right hand side of Theorem 8 gives the explicit form of $H_{p,q}(p^{-s}, q^{-s})$.

The following Theorem deals with the remaining cases.

Theorem 11 ([KK4]) *Let N_1 and N_2 be pisitive integers and $N_0 = (N_1, N_2)$. The following expression holds in $\mathrm{Re}(s) > 0$:*

$$\begin{aligned} &\zeta(s, \mathbf{F}_{p^{N_1}}) \otimes \zeta(s, \mathbf{F}_{p^{N_2}}) \\ &= \exp\left(-\frac{1}{2\pi i} \frac{N_0^2}{N_1 N_2} \sum_{n=1}^{\infty} \frac{p^{-snN_1N_2/N_0}}{n^2} + \left(\frac{isN_0 \log p}{2\pi} - 1 \right) \sum_{n=1}^{\infty} \frac{p^{-snN_1N_2/N_0}}{n} \right. \\ &\qquad\qquad \left. + \sum_{n=1}^{\infty} \frac{p^{-snN_1} f_1(n) + p^{-snN_2} f_2(n)}{n} + Q_p(s) \right), \end{aligned}$$

where $Q_p(s)$ is a quadratic polynomial in s and

$$f_1(n) = \begin{cases} (e^{2\pi i n N_1/N_2} - 1)^{-1} & (\frac{N_2}{N_0} \not| n) \\ \frac{N_2 - N_0}{2N_0} & (\frac{N_2}{N_0} | n) \end{cases},$$

$$f_2(n) = \begin{cases} (e^{2\pi i n N_2/N_1} - 1)^{-1} & (\frac{N_1}{N_0} \not| n) \\ \frac{N_1 - N_0}{2N_0} & (\frac{N_1}{N_0} | n) \end{cases}.$$

A generalization of the preceding theorems to the case of three zeta functions was recently done by Akatsuka as follows.

Theorem 12 ([A]) *Let* p, q, r *be distinct primes. In* $\mathrm{Re}(s) > 0$ *we have*

$$\begin{aligned}
&\zeta(s, \mathbf{F}_p) \otimes \zeta(s, \mathbf{F}_q) \otimes \zeta(s, \mathbf{F}_r) \\
&\quad = e^{Q(s)}(1-p^{-s})^{-\frac{1}{4}}(1-q^{-s})^{-\frac{1}{4}}(1-r^{-s})^{-\frac{1}{4}} \\
&\quad\quad \exp\Bigg(-\frac{1}{4}\sum_{n_1=1}^{\infty} \frac{\cot\left(\pi n_1 \frac{\log p}{\log q}\right)\cot\left(\pi n_1 \frac{\log p}{\log r}\right)}{n_1\ p^{n_1 s}} \\
&\quad\quad\quad -\frac{1}{4}\sum_{n_2=1}^{\infty} \frac{\cot\left(\pi n_2 \frac{\log q}{\log p}\right)\cot\left(\pi n_2 \frac{\log q}{\log r}\right)}{n_2\ q^{n_2 s}} \\
&\quad\quad\quad -\frac{1}{4}\sum_{n_3=1}^{\infty} \frac{\cot\left(\pi n_3 \frac{\log r}{\log p}\right)\cot\left(\pi n_3 \frac{\log r}{\log q}\right)}{n_3\ r^{-n_3 s}} \\
&\quad\quad\quad +\frac{i}{4}\sum_{n_1=1}^{\infty} \frac{\cot\left(\pi n_1 \frac{\log p}{\log q}\right)+\cot\left(\pi n_1 \frac{\log p}{\log r}\right)}{n_1\ p^{n_1 s}} \\
&\quad\quad\quad +\frac{i}{4}\sum_{n_2=1}^{\infty} \frac{\cot\left(\pi n_2 \frac{\log q}{\log p}\right)+\cot\left(\pi n_2 \frac{\log q}{\log r}\right)}{n_2\ q^{n_2 s}} \\
&\quad\quad\quad +\frac{i}{4}\sum_{n_3=1}^{\infty} \frac{\cot\left(\pi n_3 \frac{\log r}{\log p}\right)+\cot\left(\pi n_3 \frac{\log r}{\log q}\right)}{n_3\ r^{n_3 s}} \Bigg).
\end{aligned}$$

Here we present the outline of our proof of Theorem 7. We use the multiple sine function defined in [KK1]. We recall the definitions as follows: The multiple Hurwitz zeta function is defined by Barnes [B] as

$$\zeta_r(s, z, \underline{\omega}) = \sum_{n_1,\dots,n_r=0}^{\infty} (n_1\omega_1 + \cdots + n_r\omega_r + z)^{-s}$$

for $\underline{\omega} = (\omega_1, ..., \omega_r)$ with $\omega_j > 0$ and $\mathrm{Re}(s) > r$. The multiple gamma function is also defined as

$$\Gamma_r(z, \underline{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z, \underline{\omega})\Big|_{s=0} \right).$$

We define the multiple sine function [KK1] as

$$S_r(z, \underline{\omega}) = \Gamma_r(z, \underline{\omega})^{-1}\Gamma_r(\omega_1 + \cdots + \omega_r - z, \underline{\omega})^{(-1)^r}.$$

We put for simplicity as $S_r(z) := S_r(z,(1,...,1))$, $\Gamma_r(z) := \Gamma_r(z,(1,...,1))$, $\Gamma_1(z) = \Gamma_1(z,1) = \Gamma(z)/\sqrt{2\pi}$ and $S_1(z) = S_1(z,1) = 2\sin(\pi z)$.

Lemma 13 *The absolute tensor product in Theorem 7 is expressed as follows:*

$$(Z_1 \otimes Z_2)(s) = e^{Q(s)} S_2\left(is, \left(\frac{2\pi}{l_1}, \frac{2\pi}{l_2}\right)\right),$$

where $Q(s)$ is a polynomial of degree at most two, which depends on l_1 and l_2.

Proof. The definitions of the absolute tensor product and the multiple sine functions easily lead us to the identity.∎

Next we obtain the "Euler product" expression of the double sine function:

Lemma 14 ([KK2]) *If both $\dfrac{\omega_1}{\omega_2}$ and $\dfrac{\omega_2}{\omega_1}$ are generic and* $\mathrm{Im}(z) > 0$*,*

$$\begin{aligned} & S_2(z,(\omega_1,\omega_2)) \\ = \quad & \exp\left(\frac{1}{2i}\sum_{k=1}^{\infty}\frac{1}{k}\cot\left(\pi k\frac{\omega_2}{\omega_1}\right)e^{2\pi i k\frac{z}{\omega_1}} + \frac{1}{2i}\sum_{n=1}^{\infty}\frac{1}{n}\cot\left(\pi n\frac{\omega_1}{\omega_2}\right)e^{2\pi i n\frac{z}{\omega_2}}\right. \\ & +\frac{1}{2}\log\left(1-e^{2\pi i\frac{z}{\omega_1}}\right) + \frac{1}{2}\log\left(1-e^{2\pi i\frac{z}{\omega_2}}\right) \\ & \left. +\frac{\pi i z^2}{2\omega_1\omega_2} - \frac{\pi i}{2}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)z + \frac{\pi i}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}+3\right)\right) \end{aligned}$$

Proof. First we establish the "signatured" Poisson summation formula, counting only zeros in the upper half plane, with the test function

$$H(t) := (t-z)^{-2} - (t+z)^{-2}.$$

By Cauchy's theorem we have

$$H(k\omega_1 + n\omega_2) = \frac{1}{(2\pi i)^2}\int_C\int_C h(s_1+s_2)\frac{\xi_1'}{\xi_1}(s_1)\frac{\xi_2'}{\xi_2}(s_2)ds_1 ds_2 \qquad (1)$$

with $\xi_1(s) = \sinh\left(\dfrac{\pi s}{\omega_1}\right)$ and $\xi_2(s) = \sinh\left(\dfrac{\pi s}{\omega_2}\right)$, and where

$$C = \partial\{s \in \mathbb{C} \mid |\mathrm{Re}(s)| < \alpha,\ |s| > \alpha,\ \mathrm{Im}(s) > 0\}.$$

Calculating the integrals in the right hand side of (1) leads to the signatured Poisson summation formula:

$$\begin{aligned}\sum_{k,n>0} H(k\omega_1 + n\omega_2) &= -\frac{1}{2}\left(\sum_{k>0} H\left(k\omega_1\right) + \sum_{n>0} H\left(n\omega_2\right)\right) \\ &\quad -\frac{i}{2\omega_1}\sum_{k>0}\cot\left(\pi\frac{k\omega_2}{\omega_1}\right)\widetilde{H}\left(\frac{2\pi k}{\omega_1}\right) \\ &\quad -\frac{i}{2\omega_2}\sum_{n>0}\cot\left(\pi\frac{n\omega_1}{\omega_2}\right)\widetilde{H}\left(\frac{2\pi n}{\omega_2}\right) - \frac{i}{2}\widetilde{H}'(0). \quad (2)\end{aligned}$$

Then the left hand side of (2) is equal to

$$\frac{d^2}{ds^2}\log S_2(z,\omega_1,\omega_2).$$

Thus

$$S_2(z,\omega_1,\omega_2) = \exp\left(\iint (2)dzdz\right),$$

where we substitute (2) with its right hand side. ∎

2 Application to Special Values

By using the multiple sine function appeared in the proof of our theorems in the preceding section, we express some unknown special values for the Riemann zeta and Dirichlet L-functions.

Theorem 15 ([KK3]) *Let $0 < n, k \in \mathbb{Z}$ and put*

$$a(2n+1,k) = \sum_{l=1}^{k}(-1)^{k-l}l^{2n}\binom{2n+1}{k-l},$$

then we have

$$\zeta(2n+1) = \frac{2^{2n+1}\pi^{2n}}{(-1)^{n-1}(2n)!}\log\prod_{k=1}^{n} S_{2n+1}(k)^{a(2n+1,k)}.$$

Examples.

$$\begin{aligned}
\zeta(3) &= 4\pi^2 \log S_3(1), \\
\zeta(5) &= -\frac{4\pi^4}{3} \log(S_5(1) S_5(2)^{11}), \\
\zeta(7) &= \frac{8\pi^6}{45} \log(S_7(1) S_7(2)^{57} S_7(3)^{302}).
\end{aligned}$$

Theorem 16 ([KK3]) *Let χ be a primitive odd Dirichlet character* $\pmod N$. *Then*

$$L(2,\chi) = \frac{2\pi i \tau(\chi)}{N^2} \log \prod_{k=1}^{N-1} \left(S_2\left(\frac{k}{N}\right)^N S_1\left(\frac{k}{N}\right)^k \right)^{\bar{\chi}(k)}.$$

Examples.

$$\begin{aligned}
L(2, \left(\frac{-4}{*}\right)) &= \frac{-\pi}{4} \log \left(S_2\left(\frac{1}{4}\right)^4 S_1\left(\frac{1}{4}\right) S_2\left(\frac{3}{4}\right)^{-4} S_1\left(\frac{3}{4}\right)^{-3} \right) \\
&= \frac{\pi}{4} \log \left(2^{-3} S_2\left(\frac{1}{4}\right)^{-8} \right), \\
L(2, \left(\frac{-3}{*}\right)) &= \frac{-2\sqrt{3}\pi}{9} \log \left(S_2\left(\frac{1}{3}\right)^3 S_1\left(\frac{1}{3}\right) S_2\left(\frac{2}{3}\right)^{-3} S_1\left(\frac{2}{3}\right)^{-2} \right) \\
&= \frac{4\sqrt{3}\pi}{9} \log \left(\frac{3}{4} S_2\left(\frac{1}{3}\right)^{-3} \right).
\end{aligned}$$

Theorem 17 ([KK3]) *Let χ be a primitive even Dirichlet character* $\pmod N$.

$$L(3,\chi) = \frac{2\pi^2 \tau(\chi)}{N^3} \log \prod_{k=1}^{N-1} \left(S_3\left(\frac{k}{N}\right)^{2N^2} S_2\left(\frac{k}{N}\right)^{2Nk-3N^2} S_1\left(\frac{k}{N}\right)^{k^2} \right)^{\bar{\chi}(k)}.$$

Examples.

$$
\begin{aligned}
L(3, \left(\frac{12}{*}\right)) \quad = \quad \frac{\sqrt{3}\pi^2}{432} \log \Bigg(& S_3\left(\frac{1}{12}\right)^{288} S_2\left(\frac{1}{12}\right)^{-408} S_1\left(\frac{1}{12}\right) \\
& S_3\left(\frac{5}{12}\right)^{-288} S_2\left(\frac{5}{12}\right)^{312} S_1\left(\frac{5}{12}\right)^{-25} \\
& S_3\left(\frac{7}{12}\right)^{-288} S_2\left(\frac{7}{12}\right)^{264} S_1\left(\frac{7}{12}\right)^{-49} \\
& S_3\left(\frac{11}{12}\right)^{288} S_2\left(\frac{11}{12}\right)^{-164} S_1\left(\frac{11}{12}\right)^{121} \Bigg).
\end{aligned}
$$

3 Application to Γ-factors of Selberg Zeta Functions

Let $M = \Gamma\backslash G/K$ be a compact locally symmetric space of rank one. In this section we present the explicit form of the Γ-factors of the Selberg zeta function of M. When $\dim M$ is odd, it has only trivial Γ-factors which are exponential of some polynomials. So in what follows we assume $\dim M$ is even.

Let $M' = G'/K$ be the compact dual symmetric space of M which is given by the following table:

G	K	G'	M'
$SO(1,n)$	$SO(n)$	$SO(1+n)$	S^n
$SU(1,n)$	$SU(n)$	$SU(1+n)$	$\mathbf{P}^n_{\mathbb{C}}$
$Sp(1,n)$	$Sp(n)$	$Sp(1+n)$	$\mathbf{P}^n_{\mathbf{H}}$
F_4	$\mathrm{Spin}(9)$	F_4'	$\mathbf{P}^2_{\mathbf{O}}$

Let σ be a unitary representation of Γ. The Selberg zeta function $Z_M(s,\sigma)$ is defined by Gangolli [G]. It has an analytic continuation to all $s \in \mathbb{C}$ as a meromorphic function of order $\dim M$. It also has a functional equation:

$$Z_M(2\rho_0 - s, \sigma) = Z_M(s,\sigma) \exp\left(\mathrm{vol}(M) \dim(\sigma) \int_0^{s-\rho_0} \mu_M(it)dt \right)$$

where $\rho_0 > 0$ and $\mu_M(t)$ is the Plancherel measure.

Lemma 18 *Let $S(\Delta_{M'})$ be the set of eigenvalues of $\Delta_{M'}$. The spectral zeta function*

$$\zeta\left(s, z, \sqrt{\Delta_{M'} + \rho_0^2}\right) := \sum_{\lambda \in S(\Delta_{M'})} \left(\sqrt{\lambda + \rho_0^2} + z\right)^{-s}$$

is holomorphic at $s = 0$.

Thus we define

$$\prod_{\lambda \in S(\Delta_{M'})} \left(\sqrt{\lambda + \rho_0^2} + z\right) = \det\left(\sqrt{\Delta_{M'} + \rho_0^2} + z\right).$$

Actually

$$\det\left(\sqrt{\Delta_{M'} + \rho_0^2} + s - \rho_0\right)^{-1}$$
$$= \begin{cases} \Gamma_{2n}(s)\Gamma_{2n}(s+1) & (G = SO(1,2n)) \\ \prod_{k=0}^{n} \Gamma_{2n}(s+k)^{\binom{n}{k}^2} & (G = SU(1,n)) \\ \prod_{k=0}^{2n-1} \Gamma_{4n}(s+k)^{\frac{1}{2n}\binom{2n}{k}\binom{2n}{k+1}} & (G = Sp(1,n)) \\ \Gamma_{16}(s)\Gamma_{16}(s+1)^{10}\Gamma_{16}(s+2)^{28} \\ \quad \times\Gamma_{16}(s+3)^{28}\Gamma_{16}(s+4)^{10}\Gamma_{16}(s+5) & (G = F_4) \end{cases}$$

Theorem 19 ([KK1]) *Put*

$$\Gamma_M(s,\sigma) = \det\left(\sqrt{\Delta_{M'} + \rho_0^2} + s - \rho_0\right)^{\mathrm{vol}(M)\dim(\sigma)(-1)^{\dim M/2}}.$$

Then $\hat{Z}_M(s,\sigma) = \Gamma_M(s,\sigma)Z_M(s,\sigma)$ satisfies the symmetric functional equation:

$$\hat{Z}_M(s,\sigma) = \hat{Z}_M(2\rho_0 - s, \sigma).$$

Proof. We prove for the case of $SO(1,2n)$. All other cases are proved by similar methods. It suffices to show

$$\exp\left(\int_0^{s-\rho_0}\mu_M(it)dt\right)^{(-1)^{\frac{\dim M}{2}}} = S_{2n}(s)S_{2n}(s+1). \tag{3}$$

Both sides are equal to 1, when $s=\rho_0=n-\frac{1}{2}$. We compare the logarithmic derivative of (3). We appeal to the differential equation of $S_r(z)$ which is obtained in [KK1]:

$$\frac{S_r'}{S_r}(z) = (-1)^{r-1}\binom{z-1}{r-1}\pi\cot(\pi z).$$

Theorem follows by the facts

$$\mu_M(it) = (-1)^n P_M(t)\pi\tan(\pi t)$$

and

$$P_M(t) = \frac{2}{(2n-1)!}t\prod_{k=1}^{n-1}\left(t^2-\left(k-\frac{1}{2}\right)^2\right). \blacksquare$$

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